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't Hooft vortices and phases of large N gauge theory

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Abstract: It is shown that a pair of vortex and anti-vortex is completely screened in $2+1$ dimensional Yang-Mills theory and $3+1$ dimensional Yang-Mills theory in the strong coupling limit, based on the recent conjecture of Maldacena. This is consistent with the fact that these theories exhibit confinement.

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't Hooft suggested a disorder variable, dual to the Wilson loop, for a non-Abelian gauge theory [1]. The exchange algebra satisfied by this operator and the Wilson loop operator enables a powerful argument for possible phases of the gauge theory. Generalizing a solution to D-brane Nambu-Goto action of [2], it is proposed in [3] that a heavy 't Hooft vortex on D2-branes corresponds to the point where an orthogonal D2-brane intersects. A vortex line on a stack of D3-branes is T-dual of this intersection point, and so on.

The recent proposal of duality between a large N strongly coupled Super Yang-Mills theory and anti-de Sitter supergravity provides a powerful tool to make many predictions for SYM [4, 5, 6, 7]. In particular, the expectation value of a temporal Wilson loop can be easily computed [8, 9, 10, 11]. The spatial Wilson loop at a finite temperature exhibits the area law, indicating confinement of the lower dimensional Yang-Mills theory without supersymmetry, thereby fulfills one of the old prophecies [11, 12]. In addition, quark and anti-quark were shown to be screened at a finite temperature if the separation is sufficiently large [13, 14].

In a pure Yang-Mills theory in $2+1$ dimensions or $3+1$ dimensions, test quarks are confined, or there is a linear potential between a quark and an anti-quark. 't Hooft vortices or vortex-loops should exhibit a different behavior. It is shown in [3] that in a maximally supersymmetric Yang-Mills theory, the interaction potential between a vortex and an anti-vortex obeys a power law. We therefore expect this potential be weaker in a confining theory, such as a pure Yang-Mills theory. The purpose of this short note is to show that indeed a complete screening occurs in a confining theory, thereby to fulfill another old prophecy. The total Green's function of two vortices is equal to the disconnected part, which is given by the masses of vortices and thus displays the perimeter law in a trivial fashion.

To obtain a strongly coupled pure Yang-Mills theory in p dimensional spacetime, following Witten [11], we start with Dp-branes and wrap the Euclidean time around a circle of circumference $\beta = 1/T$. Assigning anti-periodic boundary condition to fermions gives mass of order T to fermions. Scalars receive radiative correction to their mass at one-loop, and hopefully the correction persists in the strong coupling limit. Thus, at energies much lower than T, the theory is that of effective p dimensional pure Yang-Mills theory with Euclidean spacetime. However, it is difficult to recover the weak coupling regime, due to the singularity in the near horizon geometry caused by compactification on a circle. For instance, it is not known how to directly see the dimensional transmutation in the 4 dimensional Yang-Mills theory, namely QCD, the most interesting case.

Consider D3-branes first. With a finite temperature, the classical solution in the near horizon region is that of the anti-de Sitter black hole

$$ds^2/\alpha' = \frac{U^2}{R^2}(f(U)dt^2 + dX_i^2) + \frac{R^2}{U^2}(f^{-1}(U)dU^2 + U^2 d\Omega_5), \tag{1}$$

where $i = 1, 2, 3$, $f(U) = 1 - (U_T/U)^4$. The dilaton is constant, and $R^2 = (4\pi g N)^{1/2}$.

The 4D Yang-Mills coupling $g_4^2 \sim g$. We have Wick-rotated time. For the above geometry to be non-singular at the horizon $U = U_T$, the circumference of t must be $\beta = \pi R^2/U_T$, or $U_T = \pi R^2 T$.

We are interested in the effective 3D Yang-Mills theory, whose coupling constant is given by $g_3^2 = g_4^2 T$. This sets a mass scale much smaller than T. However, the physical mass scale is set by $g_3^2 N$ which is much greater than T. In order to avoid complicating theory with KK modes, we should consider physics at distances much larger than $1/T$. We shall take X_1 as the Euclidean time of the 3D world.

An orthogonal D3-brane intersecting the source D3-branes along (t, X_1) creates a vortex in 3 dimensions. After dimensional reduction in t, the intersection is parametrized by X_1, and is regarded as the world-line of the vortex. When N is finite, the solution clearly indicates that this is a 't Hooft vortex [3]. The D3-brane can be regarded as a D2-brane after dimensional reduction along t, and has a tension $T_3\beta$. Of course this agrees with the D2-brane tension if one performs T-duality along t. We shall set $\alpha' = 1$ in the rest of the paper. T_3 measured in string unit is a pure number $c_3 = 1/[(2\pi)^3 g]$. We parametrize the other two spatial dimensions of the orthogonal D3-brane by (U, θ), where θ is embedded into the five sphere S^5. The total energy of a single D3-brane, taking X_1 as time, is given by

$$E_0 = c_3\beta \int d\theta \int_{U_T}^{\infty} U dU, \tag{2}$$

where we assume that the D3-brane stretches all the way from spatial infinity to the horizon. Notice that, although the background is not supersymmetric, and the orthogonal D3-brane is not a BPS state, all the nontrivial U dependent factors cancel and the energy is equal to that of a D3-brane embedded into a flat spacetime. The only difference is that it does not extend to the origin $U = 0$, which is the singularity of metric (1).

Before we set out to do further calculation, we want to pose and clarify a puzzle. The existence of the 't Hooft vortex proposed in [3] depends crucially on the existence of a complex Higgs field. The configuration, for a given N, is given by (A, w), where w is a complex $N \times N$ Higgs field which is singular at the location of the vortex. A is a gauge field having a nonvanishing holonomy around the vortex. Now, as we said before, in 3 dimensions the theory is effectively a pure Yang-Mills theory, so why does this vortex still exist? From the perspective of the 4D theory on D3-branes, the answer is that classically the Higgs fields are still massless, even though they gain a mass after turning on a temperature, the vortex is so heavy that it still exists. By the same token, it also exists in the 3D gauge theory as a heavy probe. The same question arises when we use the spatial Wilson loop to probe the 3D gauge theory. As shown in [2], the solution representing an open string attached to D-branes also depends on the Higgs field crucially. Although there is no massless Higgs field in the effective 3D theory, as a heavy probe the open string still exists. As argued by Witten [11], the properties of these probes depend on the gauge field configuration, when the Higgs fields are given a mass. Also, the exchange algebra generated by the Wilson loop and the 't Hooft vortex depends on the gauge field only [1].

Consider a pair of orthogonal D3-brane and anti-D3-brane. This system should be regarded as a single D3-brane. The simplest topology is that given in fig. 1, where a throat connects the two D3-branes. When $T=0$, as shown in [3], there is always a saddle point of the Nambu-Born-Infeld action with this topology. The energy of this saddle point is smaller than that of two disconnected branes, thus it is more energetically favored. The difference in energy is then interpreted as the interaction energy. For $T>0$, no matter how small T is, we shall show that for a separation L large than a certain value L_{max}, which is order β, there does not exist a saddle point with the topology in fig. 1. There is another scale $L_c < L_{max}$, also of order β, at which the energy of this configuration is equal to that of the bare energy, and in between the two points, the "interaction energy" is positive, therefore the connected configuration is not favored. We conclude that for a separation $L > L_c$, the interaction energy vanishes. Interpreted in terms of the 3D gauge theory, the vortex and the anti-vortex are completely screened. As we shall explain later, this answer can not be the exact one.

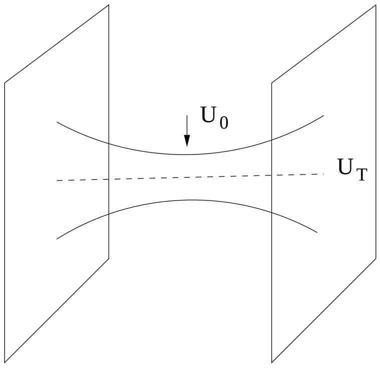

Figure 1: A D3-brane and an anti-D3-brane connected by a throat. In the Euclidean metric, $U=U_T$ is the origin.

Let the two D3-branes be separated in direction X_2 by a distance L, for simplicity use x to denote X_2. The configuration in fig. 1 is symmetric with respect to $x=0$, thus U is an even function of x. The induced world-volume metric on the D3-brane is

$$\begin{aligned} h_{tt} &= \frac{U^2}{R^2}f(U), \quad h_{11} = \frac{U^2}{R^2}, \\ h_{\theta\theta} &= R^2, \quad h_{xx} = \frac{U^2}{R^2} + \frac{R^2}{U^2}f^{-1}(U)(U_{,x})^2, \end{aligned} \tag{3}$$

and the energy is given by

$$E = 4\pi\beta c_3 \int_0^{L/2} dx \left(\frac{U^6}{R^4}f(U) + U^2(U_{,x})^2\right)^{\frac{1}{2}}. \tag{4}$$

The classical solution minimizing (4)

$$x = \frac{R^2}{U_0}\int_1^{U/U_0} dy (y^4-\lambda)^{-\frac{1}{2}}\left(\frac{y^2(y^4-\lambda)}{1-\lambda} - 1\right)^{-\frac{1}{2}}, \tag{5}$$

where $\lambda = (U_T/U_0)^4$. Given a value for λ, L is determined by

$$L = \frac{2\beta}{\pi}\lambda^{\frac{1}{4}}(1-\lambda)^{\frac{1}{2}}\int_1^{\infty} dy (y^4-\lambda)^{-\frac{1}{2}}(y^6-\lambda y^2 - 1+\lambda)^{-\frac{1}{2}}, \tag{6}$$

where we used the relation $U_T = \pi R^2 T$.

The interaction energy is obtained by subtracting from (4) twice of (2)

$$V = 4\pi^3 c_3 R^4 T\lambda^{-\frac{1}{2}}\left(\int_1^\infty dy[y^2(y^4-\lambda)^{\frac{1}{2}}(y^6-\lambda y^2-1+\lambda)^{-\frac{1}{2}}-y]-\frac{1}{2}(1-\sqrt{\lambda})\right). \quad (7)$$

For a sufficiently small L, the interaction approaches that in the zero temperature case, except that here the interpretation is in 3D gauge theory rather than in the 4D SYM. V is independent of g. The small L behavior is

$$V = -NTL^{-2}(c_1 + c_2(LT)^4 + \ldots), \quad (8)$$

both c_1 and c_2 are positive constant. This is to be contrasted to the zero temperature interaction energy in the 3D SYM, where the power law is $L^{-4/3}$. Of course with small L we are probing the 4D SYM at finite temperature, not the 3D pure gauge theory.

It is seen from (6) that not for all L there is a solution. A upper bound for the integral in (6) is $\lambda^{-1/2}\ln[(1+\sqrt{\lambda})/\sqrt{1-\lambda}]$. Combined with the additional factor $\lambda^{\frac{1}{4}}(1-\lambda)^{\frac{1}{2}}$ it has a finite maximal value. Denote the maximal value of L by L_{max} and the corresponding λ by λ_{max}. A numerical calculation shows $L_{max} = 0.22\beta$, $\lambda_{max} = 0.62$. Thus, at least beyond L_{max} there is no saddle point solution of topology in fig. 1. We conclude that the interaction energy vanishes, and vortices are completely screened.

However, the interaction rises to 0 at the value $\lambda_c = 0.28$ (our boundary condition is $\lambda = 0$, $L = 0$), and the corresponding separation is $L_c = 0.196\beta < L_{max}$. Beyond L_c, the interaction energy becomes positive and one must discard the saddle point. We then see that for $L > L_c$, the vortices are screened. This value is comparable to the compactification scale β. Indeed for this separation the theory is not effectively 3 dimensional yet.

In the above calculation, the only constraint on various parameters is $g_4^2 N \gg 1$. The three dimensional effective Yang-Mills coupling is $g_3^2 N = g_4^2 NT$. We are exploring the infrared regime of the 3 dimensional theory if the separation between the vortex and the anti-vortex satisfies $g_4^2 N(LT) \gg 1$. The critical value L_c certainly satisfies this condition. Indeed, there is a range of L smaller than L_c also satisfies this condition. But since now L is smaller than the compactification scale, we are no longer probing the 3 dimensional theory.

In the infrared regime, the 3 dimensional pure Yang-Mills theory is believed to be confining. The test vortex is dual to a test quark in the sense of [1], thus we expect screening. Our calculation however does not imply that the interaction is exactly zero. In the maximal SYM, the interaction falls off in L as a power. The vortex is coupled to an operator of color singlet, this power law can be accounted for in principle. In a confining theory such as the pure Yang-Mills, one expects that the exchange of color singlet states contributes to the interaction, thus the interaction should fall off exponentially with L, as there is a mass gap. As pointed out by Gross, such behavior can rise in a quantum calculation where a saddle point does not exist. Although the screening is perfectly understandable in a confining phase, it is also interesting to ask precisely what screens the vortices, causing the phenomenon similar to Debye screening,

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in a pure gauge theory. It is difficult to construct a microscopic vortex state in the continuum formulation, however, it is possible to do so on a lattice [15]. Perhaps it is not far off the track to speculate that the confining vacuum can be regarded as a medium of virtual vortices.

We proceed to the case of D4-branes. We obtain the 4D pure Yang-Mills theory by compactifying the Euclidean time. This is the most interesting case, since here we are making contact with QCD, although what we can say is about the strong coupling phase. D4-branes can be obtained by wrapping M5-branes around a circle, so in a sense the theory can be derived from the conformally invariant one on M5-branes. There are two scales breaking conformal invariance. The first circle, the M-circle, introduces a spacetime dependent dilaton. The second circle, the Euclidean time, introduces a scale below which a pure Yang-Mills theory is obtained.

The near horizon geometry of D4-branes at zero temperature was discussed in [16], and the finite temperature case was considered in [11, 12]. The metric and dilaton with $\alpha' = 1$ read

$$\begin{aligned} ds^2 &= \frac{U^{\frac{3}{2}}}{a}(f(U)dt^2 + dX_i^2) + aU^{-\frac{3}{2}}(f^{-1}(U)dU^2 + U^2 d\Omega_4^2), \\ e^\phi &= \frac{a^{3/2}U^{3/4}}{2\pi^{3/2}}N^{-1}, \end{aligned} \tag{9}$$

where $i = 1, \dots, 4$, $f(U) = 1-(U_T/U)^3$ and $a = g_{YM}\sqrt{N}/(2\sqrt{\pi})$. g_{YM}^2 is the Yang-Mills coupling on D4-branes. Written in the above form, the metric is independent of N, and the string coupling scales with N as N^{-1}, clearly indicating that this is an attractive framework for dealing with QCD strings. The relation between the circumference of t and U_T is $\sqrt{U_T} = 4\pi a/(3\beta)$.

To trust the classical supergravity, the value of U must lie in between $1/(g_{YM}^2 N) \ll U \ll N^{1/3}/g_{YM}^2$ [12]. U_T has to satisfy the same conditions. Since the effective 4D gauge theory coupling is given by $g_4^2 = g_{YM}^2 T$, this puts constraints on the 4D coupling: $1 \ll g_4^2 N \ll N^{2/3}$. Thus the 4D pure gauge theory is in the strong coupling regime. In particular, the geometry (9) can not be used to describe the asymptotic region, where the importance of the dimensional transmutation begins to appear.

An orthogonal D4-brane intersects source branes along (t, X_1, X_2). After dimensional reduction along t, and taking X_1 as the Euclidean time in the effective 4D theory, X_2 describes a vortex line, or a 't Hooft loop. For a D4-brane stretching all the way to the horizon, the bare energy per unit length along X_2 is

$$E_0 = \frac{2\pi^{3/2}N\beta c_4}{a^2}\int d\theta \int_{U_T}^\infty dU, \tag{10}$$

where $c_4 = 1/(2\pi)^4$. Note that the factor N cancels with the one implicit in a^2.

A pair of D4-brane and anti-D4-brane again should be regarded as a single D4-brane as in fig. 1. Let the separation along $x = X_3$ be L. The total energy per unit length along X_2 of the configuration is

$$E = \frac{4\pi^{5/2}N\beta c_4}{a^2}\int dx U\left(\frac{U^3}{a^2}f + (U_{,x})^2\right)^{1/2}. \tag{11}$$

Again the solution minimizing the above energy satisfies an equation similar to (5), and the relation between L and U_0 is

$$L = \frac{3\beta}{2\pi}\lambda^{\frac{1}{6}}(1-\lambda)^{\frac{1}{2}}\int_1^\infty dy(y^3-\lambda)^{-\frac{1}{2}}(y^5-\lambda y^2-1+\lambda)^{-\frac{1}{2}}, \tag{12}$$

where we used the relation between T and U_T, and $\lambda = (U_T/U_0)^3$.

Similar to the case studied earlier, there is a finite maximum for L. Its numerical value is $L_{max} = 0.21\beta$, and the corresponding $\lambda_{max} = 0.64$. Thus, we have a completely screening for $L > L_{max}$.

The interaction energy for a L allowed by (12) is

$$V = 2^{-1}\pi^{-3/2}N\beta U_0^2 a^{-2}\left(\int_1^\infty dy[y^2(y^3-\lambda)^{\frac{1}{2}}(y^5-\lambda y^2-1+\lambda)^{-\frac{1}{2}} - y] - \frac{1}{2}(1-\lambda^{2/3})\right). \tag{13}$$

For small separation, the interaction potential displays a power law with an exponent different from the zero temperature case computed in [3]. This is not surprising, since here the theory is effectively 5 dimensional rather than 4 dimensional. The interaction remains negative for small λ until it reaches the value $\lambda_c = 0.23$. This gives the separation, according to (12), $L_c = 0.184\beta$. This is smaller than L_{max}. For $L > L_c$, the interaction energy becomes positive, and the saddle point solution is not the favored configuration. Again both L_{max} and L_c are comparable to β, and we can safely conclude that for a separation much larger than the compactification scale, where the theory is effectively 4 dimensional, the vortex lines are screened.

The result of the fairly straightforward calculations in this note is compatible with both the idea that an orthogonal D-brane represents a heavy 't Hooft vortex, and the idea that the large N gauge theory in the strong coupling limit can be described by classical supergravity in the near horizon geometry. It would be very interesting to study the behavior of vortices in a Higgs phase. To do this, some matter fields are to be introduced, in order to higgs the vector fields. We leave this problem for future.

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